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Generalized helical Beltrami flows in hydrodynamics and magnetohydrodynamics

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The nonlinear, three-dimensional Euler equations can be reduced to a simple linear equation when the flow has helical symmetry and when the flow consists of a rigidly rotating basic part plus a Beltrami disturbance part (with vorticity proportional to velocity or a slight generalization of this). Solutions to this linear equation represent steadily rotating, non-axisymmetric waves of arbitrary amplitude. Exact solutions can be constructed in the case of flow in a straight pipe of circular cross-section. Analogous results are obtained for the incompressible, non-dissipative equations of magnetohydrodynamics. In addition to a rigidly rotating basic flow, there may exist a toroidal magnetic field varying linearly with radius.

1. Introduction

More than a century ago, Lord Kelvin (1880) examined the linear stability of rigidly rotating flow in a pipe. He found that it was possible to construct the eigenfunctions explicitly in terms of Bessel functions, and also provided a transcendental equation for the eigenvalues. This equation showed that all disturbances are neutrally stable.

One of the results of this paper is that Kelvin's eigenfunctions turn out to be exact solutions to the full, three-dimensional Euler equations. That is, disturbances of arbitrary amplitude, and exactly the same structure and frequency as Kelvin's, solve the nonlinear Euler equations.

Exact solutions, which are steadily rotating and of permanent form, can be constructed in very general pipe geometries. The only limitation is that both the pipe geometry and the enclosed flow must have helical symmetry – for instance the pipe could have the shape of a corkscrew.

These exact solutions are constructed from a simple, linear eigen-equation, which results from assuming helical symmetry and from decomposing the flow into a basic part in rigid rotation and a disturbance part satisfying a certain relationship between its vorticity and its velocity. In the simplest case, the vorticity is proportional to velocity, a condition referred to as 'Beltrami'. But this may be generalized to requiring that the cross-product of the vorticity and the velocity be the gradient of a pressure-like quantity. Notably, the combined flow (basic plus disturbance) need not satisfy this constraint.

Analogous results apply if we add a basic toroidal magnetic field varying linearly with radius, corresponding to a uniform axial current density.

In the following section, the linear governing equation is derived and applied to the problem first investigated by Kelvin. In §3, analogous results are derived for magnetohydrodynamics. Section 4 takes a brief look at the stability of some of these solutions, by direct numerical integration, and, in many cases, it is found that an explosive growth of vorticity occurs. Section 5 summarizes the results and outlines some possible extensions and applications. Remarks on the boundary conditions for arbitrary helical domains and on more general basic flows are reserved for Appendices A and B.

2. Kelvin's problem of rigid rotation

Consider the motion of an inviscid, incompressible fluid in an infinite circular cylinder. Let a be the radius of the cylinder, and let there be a basic flow in rigid rotation about the axis of the cylinder at the rate Ω_0. A rotating frame of reference is adopted in which the basic flow is at rest.

Assume that the flow has helical symmetry (see Park, Monticello & White 1984; Landman 1990 and references therein). This means that the velocity, vorticity, and pressure fields do not vary in the vector direction $\boldsymbol{h}$. $\boldsymbol{h}$ is referred to as the 'Beltrami vector' and, in cylindrical coordinates (r, θ, z), it is defined by

$$\boldsymbol{h} = h^2(\boldsymbol{e}_z - \epsilon r \boldsymbol{e}_\theta) \tag{1}$$

with $h^2 = (1+\epsilon^2 r^2)^{-1}$ and ϵ referred to as the 'pitch' (note: $\boldsymbol{h}$ is not a unit vector). When $\epsilon = 0$, the flow is two-dimensional, and when $\epsilon = \infty$, it is axisymmetric. $\boldsymbol{h}$ is orthogonal to the radial unit vector $\boldsymbol{e}_r$, and the cross product of $\boldsymbol{h}$ with $\boldsymbol{e}_r$ defines a third orthogonal vector in the direction of $\theta + \epsilon z \equiv \phi$:

$$\boldsymbol{e}_\phi = h^{-1}\boldsymbol{h}\times\boldsymbol{e}_r = h(\boldsymbol{e}_\theta + \epsilon r \boldsymbol{e}_z), \tag{2}$$

Helical symmetry means that $\boldsymbol{h}\cdot\nabla$ applied to any scalar function of r, ϕ, and time t is zero.

Helical symmetry permits the following decomposition of the velocity and vorticity fields:

$$\boldsymbol{u} = \boldsymbol{h}\times\nabla\psi + \boldsymbol{h}v, \tag{3a}$$

$$\boldsymbol{\omega} = \boldsymbol{h}\times\nabla\chi + \boldsymbol{h}\zeta. \tag{3b}$$

This decomposition automatically satisfies $\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{\omega} = 0$ (one must use $\nabla\cdot\boldsymbol{h} = 0$ and $\nabla\times\boldsymbol{h} = -2\epsilon h^2\boldsymbol{h}$ to verify this). Now, since $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$, we have $\chi = -v$ and

$$\mathscr{L}\psi \equiv \frac{1}{rh^2}\frac{\partial}{\partial r}\left(rh^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2h^2}\frac{\partial^2\psi}{\partial\phi^2} = \zeta + 2\epsilon h^2 v. \tag{4}$$

(The latter follows by direct manipulation, noting that $\nabla = \boldsymbol{e}_r\,\partial/\partial r + \boldsymbol{e}_\phi (rh)^{-1}\,\partial/\partial\phi$.) One can continue along these lines to obtain the nonlinear evolution equations for v and ζ, but this is unnecessary at this stage. Instead, we impose the Beltrami condition

$$\boldsymbol{\omega} = -\alpha\boldsymbol{u}, \tag{5}$$

where α is a constant. Equating (3b) with $-\alpha$ times (3a) gives

$$v = \alpha\psi, \quad \zeta = -\alpha v = -\alpha^2\psi. \tag{6}$$

Hence (4) becomes a single equation for ψ,

$$\mathscr{L}\psi + (\alpha^2 - 2\epsilon\alpha h^2)\,\psi = 0. \tag{7}$$

The boundary conditions are given below.

No dynamics has been taken into account yet. For this, we turn to the full momentum and vorticity equations:

$$\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{\omega}\times\boldsymbol{u}+2\boldsymbol{\Omega}_0\times\boldsymbol{u} = -\nabla\left(\frac{p}{\rho}+\tfrac{1}{2}|\boldsymbol{u}|^2-\tfrac{1}{2}\Omega_0^2 r^2\right), \tag{8a}$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t}+\nabla\times(\boldsymbol{\omega}\times\boldsymbol{u}) = 2\boldsymbol{\Omega}_0\cdot\nabla\boldsymbol{u}, \tag{8b}$$

where $\boldsymbol{\Omega}_0 = \Omega_0 \boldsymbol{e}_z$, p is the pressure, and ρ is the uniform density. We next use $\boldsymbol{\omega}\times\boldsymbol{u} = 0$ and $\boldsymbol{h}\cdot\nabla f = 0$ for any $f(r,\phi,t)$. Taking the scalar product of both equations with $\boldsymbol{h}$ and using (3a) and (3b), we find

$$h^2\frac{\partial v}{\partial t}+2\Omega_0\boldsymbol{h}\cdot(\boldsymbol{e}_z\times\boldsymbol{h}\times\nabla\psi) = 0,$$

$$h^2\frac{\partial \zeta}{\partial t} = 2\Omega_0\,\boldsymbol{h}\cdot(\boldsymbol{e}_z\cdot\nabla(\boldsymbol{h}v)),$$

or

$$\frac{\partial v}{\partial t}+2\epsilon\Omega_0\frac{\partial \psi}{\partial \phi} = 0, \tag{9a}$$

$$\frac{\partial \zeta}{\partial t}-2\epsilon\Omega_0\frac{\partial v}{\partial \phi} = 0. \tag{9b}$$

From (6), these two equations are identical. They both yield

$$\frac{\partial \psi}{\partial t}+\frac{2\epsilon\Omega_0}{\alpha}\frac{\partial \psi}{\partial \phi} = 0, \tag{10}$$

whose general solution is $\psi(r,\varphi)$ with $\varphi = \phi-(2\epsilon\Omega_0/\alpha)\,t$, or just a steadily rotating wave. All the dynamics are encapsulated in the time-shifted angular coordinate φ. The solution to the problem then comes solely from (7), with ϕ replaced by φ (see (4)).

To get the boundary conditions for ψ in (7), note that (3a) implies that the radial, tangential, and axial velocity components in the cylindrical coordinate system are given by

$$u_r = -\frac{1}{r}\frac{\partial \psi}{\partial \phi}, \tag{11a}$$

$$u_\theta = h^2\left(\frac{\partial \psi}{\partial r}-\epsilon r v\right), \tag{11b}$$

$$u_z = h^2\left(v+\epsilon r\frac{\partial \psi}{\partial r}\right). \tag{11c}$$

Requiring that the radial flow be finite at the origin and that there be no flow through the pipe wall, we have

$$\frac{\partial \psi}{\partial \phi} = 0 \quad \text{at} \quad r = 0, a. \tag{12}$$

Substituting φ for ϕ, (7) simply becomes an eigenproblem for $\psi(r,\varphi)$. α is the eigenvalue. The exact solutions to (7) were found by realizing that they are identical to the linear solutions worked out by Kelvin. Hence, we immediately have

$$\psi(r,\varphi) = \mathscr{A}\hat{\psi}(r)\,\mathrm{e}^{\mathrm{i}m\varphi}+\mathrm{c.c.} \tag{13}$$

where $\mathscr{A}$ is an arbitrary complex constant, and

$$\hat{\psi}(r) = J_m(\mu r) - \frac{\epsilon \mu r}{\alpha} J'_m(\mu r), \tag{14}$$

where $\mu = (\alpha^2 - m^2\epsilon^2)^{\frac{1}{2}}$, and α is determined from the transcendental equation

$$J_m(\mu a) = \frac{\epsilon \mu a}{\alpha} J'_m(\mu a). \tag{15}$$

For example, when $\epsilon = 0$, $\alpha a = \pm j_{mn}$ $(J_m(j_{mn}) = 0)$, and when $\epsilon \to \infty$, $\alpha/\epsilon \to \pm m$. The corresponding velocity components, $u_r = \mathscr{A}\hat{u}_r(r)\exp(\mathrm{i}m\varphi) + \mathrm{c.c.}$, etc., using (6) and (11) are

$$\hat{u}_r = -\mathrm{i}m\left(\frac{1}{r}J_m(\mu r) - \frac{\epsilon\mu}{\alpha}J'_m(\mu r)\right), \tag{16a}$$

$$\hat{u}_\theta = \mu J'_m(\mu r) - \frac{\epsilon m^2}{\alpha r} J_m(\mu r), \tag{16b}$$

$$\hat{u}_z = \frac{\mu^2}{\alpha} J_m(\mu r). \tag{16c}$$

These are exact, nonlinear solutions to the three-dimensional Euler equations.

It is also possible to obtain exact solutions for the flow between two concentric circular cylinders. These involve both J_m and the complementary Bessel function Y_m.

If two or more solutions to (7) share a common eigenvalue α, then an arbitrary superposition of these solutions is also an exact nonlinear solution of Euler equations. For Kelvin's problem, there is in fact always a second solution – an axisymmetric steady flow – given by (14) with $m = 0$, or

$$\bar{\psi}(r) = J_0(\alpha r) + \epsilon r J_1(\alpha r), \tag{17}$$

whose corresponding velocity field is

$$\bar{u}_r = 0, \quad \bar{u}_\theta = -\alpha J_1(\alpha r), \quad \bar{u}_z = \alpha J_0(\alpha r). \tag{18}$$

(For flow between two concentric circular cylinders, there is yet a third solution, also an axisymmetric steady flow, involving the Bessel functions Y_0 and Y_1.) It is remarkable that the addition of this non-trivial axisymmetric flow, at any amplitude whatsoever, has no effect on either the temporal behaviour or the spatial structure of the non-axisymmetric part of the disturbance (16).

A slightly more general problem arises if we relax the Beltrami condition (5) to

$$\boldsymbol{\omega} \times \boldsymbol{u} = -\boldsymbol{\nabla}\pi, \tag{19}$$

where π is some scalar function (M. R. E. Proctor, personal communication). Using the expressions (3a) and (3b) for $\boldsymbol{u}$ and $\boldsymbol{\omega}$ and $\chi = -v$, (19) implies

$$[\boldsymbol{h}\cdot(\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}v)]\boldsymbol{h} - h^2(\zeta\boldsymbol{\nabla}\psi + v\boldsymbol{\nabla}v) = -\boldsymbol{\nabla}\pi;$$

hence, since $\boldsymbol{\nabla}\psi$ is perpendicular to $\boldsymbol{h}$, we must have both $v = v(\psi)$, to eliminate the component orthogonal to $\boldsymbol{\nabla}\psi$, and $\pi = \pi(\psi)$, to ensure that $\boldsymbol{\nabla}\pi$ is parallel to $\boldsymbol{\nabla}\psi$. then (19) implies

$$\zeta = -vv' + \pi'/h^2, \tag{20}$$

where a prime stands for functional differentiation with respect to ψ.

We next have to satisfy ($9a, b$), which still apply when $\pi \neq 0$. Using $v = v(\psi)$ and ζ from (20), ($9a, b$) become

$$v' \frac{\partial \psi}{\partial t} + 2\epsilon \Omega_0 \frac{\partial \psi}{\partial \phi} = 0,$$

$$\left(\frac{\pi''}{h^2} - vv'' - v'^2 \right) \frac{\partial \psi}{\partial t} - 2\epsilon \Omega_0 v' \frac{\partial \psi}{\partial \phi} = 0.$$

Multiplying the first equation by $-v'$ and equating the coefficients of $\partial \psi / \partial t$, we find

$$\pi''/h^2 - vv'' = 0,$$

or, since this must be true for arbitrary ψ, we conclude $\pi'' = v'' = 0$. Hence v' and π' are constants – that is v and π are linear functionals of ψ – and once again we obtain the linear equation (10).

Without loss of generality, we may take $v = \alpha \psi$ and $\pi = \pi' \psi$. Equation (20) then shows that

$$\zeta = -\alpha^2 \psi + \pi'/h^2, \tag{21}$$

differing from the expression in (6) by a term proportional to $1 + \epsilon^2 r^2$. Substituting these forms for ζ and v into (4), we finally obtain

$$\mathscr{L}\psi + (\alpha^2 - 2\epsilon \alpha h^2)\psi = \pi'/h^2. \tag{22}$$

Thus, relaxing the Beltrami constraint has the single effect of changing (7) to an inhomogeneous eigenvalue problem.

For Kelvin's problem, the solutions to (22) are given by the solutions to the homogeneous problem, (14) and (17), plus an additional axisymmetric, steady flow $\breve{\psi}(r)$. This additional flow is necessarily axisymmetric on account of the boundary conditions (12) and the ϕ-independent form of the inhomogeneous term in (22). One can verify that

$$\breve{\psi}(r) = \frac{\pi'}{\alpha^2}\left(1 + \frac{2\epsilon}{\alpha} + \epsilon^2 r^2\right) \tag{23}$$

does indeed satisfy (22). The corresponding velocity field is

$$\breve{u}_r = 0, \quad \breve{u}_\theta = -\frac{\pi' \epsilon r}{\alpha}, \quad \breve{u}_z = \frac{\pi'}{\alpha^2}(\alpha + 2\epsilon),$$

which is simply a rigid rotation plus a uniform axial flow. The most such a flow could be expected to do is alter the rotational frequencies of the disturbances, but it does not do even this. The more general condition (19), therefore, adds little freedom to the solutions.

3. Results for magnetohydrodynamics

The equations of magnetohydrodynamics (MHD) can similarly be reduced to a linear equation when the basic flow is sufficiently simple and when the disturbance flow is Beltrami.

Let $(\mu_0 \rho)^{\frac{1}{2}} \boldsymbol{B}$ be the magnetic field. Here μ_0 is the magnetic permeability and ρ is the (uniform) fluid density. The current density is given by $(\mu_0 \rho)^{\frac{1}{2}} \boldsymbol{j}$, with $\boldsymbol{j} = \nabla \times \boldsymbol{B}$. The governing equations of incompressible, non-diffusive MHD can be written

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{u} + \boldsymbol{B} \times \boldsymbol{j} = -\nabla \left(\frac{p}{\rho} + \tfrac{1}{2}|\boldsymbol{u}|^2 + \tfrac{1}{2}|\boldsymbol{B}|^2 \right) \equiv -\nabla \Pi, \tag{24a}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times (\boldsymbol{B} \times \boldsymbol{u}) = 0 \tag{24b}$$

(see e.g. Craik 1988; Moffatt 1989). We split the velocity and magnetic fields into a basic part and a Beltrami disturbance part:

$$\boldsymbol{u} \rightarrow \Omega_0 r\boldsymbol{e}_\theta + \boldsymbol{u}; \quad \boldsymbol{\omega} \rightarrow 2\Omega_0 \boldsymbol{e}_z + \boldsymbol{\omega}, \tag{25a}$$

$$\boldsymbol{B} \rightarrow Q_0 r\boldsymbol{e}_\theta + \boldsymbol{B}; \quad \boldsymbol{j} \rightarrow 2Q_0 \boldsymbol{e}_z + \boldsymbol{j}, \tag{25b}$$

with

$$\boldsymbol{\omega} = -\alpha\boldsymbol{u} - \beta\boldsymbol{B}, \tag{26a}$$

$$\boldsymbol{j} = -\gamma\boldsymbol{u} - \delta\boldsymbol{B}, \tag{26b}$$

$$\boldsymbol{B} = c\boldsymbol{u}, \tag{26c}$$

where all of the coefficients are constant. Taking the curl of (26c) and using (26a) and (26b), we get

$$\gamma = \alpha c, \quad \delta = \beta c.$$

We obtain one more condition on the coefficients by eliminating the disturbance nonlinearity from the left-hand side of (24a); this gives

$$\beta = -\gamma.$$

Hence, defining

$$\tilde{\alpha} \equiv \alpha + \beta c = \alpha(1-c^2), \tag{27}$$

we see from (26a,b,c) that

$$\boldsymbol{\omega} = -\tilde{\alpha}\boldsymbol{u}, \quad \boldsymbol{j} = c\boldsymbol{\omega}. \tag{28}$$

Decomposing $\boldsymbol{u}$ and $\boldsymbol{\omega}$ into helical variables as before (see (3a,b) and (4)), we obtain (6) and (7) with $\tilde{\alpha}$ in place of α. The spatial structure equation is therefore the same as in the case of the Euler equations.†

In order to get the temporal behaviour of the solution, we must work with (24a,b) (after substituting the fields given in (25)). The Beltrami constraints reduce (24a,b) to

$$\frac{\partial \boldsymbol{u}}{\partial t} + 2\Omega_0 \boldsymbol{e}_z \times \boldsymbol{u} + \Omega_0 \boldsymbol{\omega} \times r\boldsymbol{e}_\theta + Q_0 r\boldsymbol{e}_\theta \times \boldsymbol{j} + 2Q_0 \boldsymbol{B} \times \boldsymbol{e}_z = -\boldsymbol{\nabla}\Pi, \tag{29a}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + Q_0 \boldsymbol{\nabla} \times (r\boldsymbol{e}_\theta \times \boldsymbol{u}) + \Omega_0 \boldsymbol{\nabla} \times (\boldsymbol{B} \times r\boldsymbol{e}_\theta) = 0. \tag{29b}$$

We obtain the evolution equation for ψ by taking the scalar product of (29a,b) and their curls with $\boldsymbol{h}$. All these equations must give the same evolution for ψ. We make use of the following results for any pair of vectors $\boldsymbol{u}$ and $\boldsymbol{\omega}$ decomposed into helical variables ($\boldsymbol{u}$ and $\boldsymbol{\omega}$ satisfy (3a,b) and (4)):

$$\left.\begin{aligned}
h^{-2}\boldsymbol{h}\cdot(\boldsymbol{e}_z \times \boldsymbol{u}) &= \epsilon\frac{\partial \psi}{\partial \phi}, \\
h^{-2}\boldsymbol{h}\cdot(\boldsymbol{u} \times r\boldsymbol{e}_\theta) &= -\frac{\partial \psi}{\partial \phi}, \\
h^{-2}\boldsymbol{h}\cdot[\boldsymbol{\nabla} \times (\boldsymbol{e}_z \times \boldsymbol{u})] &= -\epsilon\frac{\partial v}{\partial \phi}, \\
h^{-2}\boldsymbol{h}\cdot[\boldsymbol{\nabla} \times (\boldsymbol{u} \times r\boldsymbol{e}_\theta)] &= \frac{\partial v}{\partial \phi}, \\
h^{-2}\boldsymbol{h}\cdot\{\boldsymbol{\nabla} \times [\boldsymbol{\nabla} \times (\boldsymbol{u} \times r\boldsymbol{e}_\theta)]\} &= \frac{\partial \zeta}{\partial \phi}.
\end{aligned}\right\} \tag{30}$$

† We similarly obtain the inhomogeneous equation (22) in place of (7) if we relax the Beltrami conditions (26) to $\boldsymbol{\omega} \times \boldsymbol{u} + \boldsymbol{B} \times \boldsymbol{j} = -\boldsymbol{\nabla}\pi$ and $\boldsymbol{B} \times \boldsymbol{u} = \boldsymbol{\nabla}\Gamma$ (see end of §2). Consistency requires $\Gamma =$ constant, which implies $\boldsymbol{B} = c\boldsymbol{u}$, and $\pi = \pi'\psi$, where π' is a specified constant.

Then, using $\boldsymbol{B} = c\boldsymbol{u}$, $v = \tilde{\alpha}\psi$ and $\zeta = -\tilde{\alpha}^2\psi$, (29$a$) and its curl both ultimately reduce to

$$\frac{\partial \psi}{\partial t} + (\Omega_0 - cQ_0)\left(1 + \frac{2\epsilon}{\tilde{\alpha}}\right)\frac{\partial \psi}{\partial \phi} = 0, \tag{31}$$

while (29b) and its curl both reduce to

$$\frac{\partial \psi}{\partial t} + \left(\Omega_0 - \frac{Q_0}{c}\right)\frac{\partial \psi}{\partial \phi} = 0. \tag{32}$$

Consistency then requires that the coefficients of $\partial\psi/\partial\phi$ match, and from this we obtain c for a prescribed flow (Ω_0, Q_0):

$$c = \left\{\frac{\epsilon\Omega_0}{\tilde{\alpha}Q_0} \pm \left[\left(\frac{\epsilon\Omega_0}{\tilde{\alpha}Q_0}\right)^2 + 1 + \frac{2\epsilon}{\tilde{\alpha}}\right]^{\frac{1}{2}}\right\}\bigg/\left(1 + \frac{2\epsilon}{\tilde{\alpha}}\right). \tag{33}$$

There are two roots, implying that there are two disturbance magnetic field strengths consistent with linear evolution (for a given eigenvalue $\tilde{\alpha}$). When the pitch $\epsilon = 0$, $c = \pm 1$, so the solutions correspond to 'Elsasser' solutions (see Moffatt 1989). When $\epsilon = \infty$, the two roots are $c = 0$ and $c = \Omega_0/Q_0$.

In summary, the inclusion of a magnetic field changes only the frequency of rotation of the solutions. The spatial structure of the solutions is still determined from the eigen-equation (7) or (22), the solutions to which are unchanged by the presence of a magnetic field.

4. A few remarks on stability

In this section, direct numerical simulations are used to examine the interaction of two superposed exact solutions, as a first, qualitative look at the stability of these solutions. The calculations are performed assuming small pitch, $\epsilon \leqslant 1$, or flows which twist gradually with height. The case of small pitch is interesting not only because the resulting governing equations are simple, but because it is possible to compute on a long timescale $\tau = \epsilon t$, and the results apply irrespective of an enclosing cylindrical boundary. That is, one obtains the results for a contained and a free vortex simultaneously. One drawback is that the results apply only to weakly nonlinear situations, when the vorticity is only slightly perturbed from its uniform value in rigid rotation.

We first derive the governing equations for small pitch. Assume for now that the flow is contained within a cylinder of unit radius, so that the pitch ϵ is a dimensionless small parameter. Beginning with the full three-dimensional Euler equations in cylindrical coordinates (r, θ, z), with density $\rho = 1$,

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} + \frac{\partial p}{\partial r} = 0, \tag{34a}$$

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} + \frac{1}{r}\frac{\partial p}{\partial \theta} = 0, \tag{34b}$$

$$\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z} + \frac{\partial p}{\partial z} = 0, \tag{34c}$$

$$\frac{\partial (r u_r)}{\partial r} + \frac{\partial u_\theta}{\partial \theta} + r\frac{\partial u_z}{\partial z} = 0, \tag{34d}$$

suppose we try the expansion

$$\begin{aligned} u_r &= \qquad \epsilon u_{r1}(r,\varphi,\tau)+\epsilon^2 u_{r2}(r,\varphi,\tau)+\dots,\\ u_\theta &= \tfrac{1}{2}r+\epsilon u_{\theta 1}(r,\varphi,\tau)+\epsilon^2 u_{\theta 2}(r.\varphi,\tau)+\dots,\\ u_z &= \qquad \epsilon u_{z1}(r,\varphi,\tau)+\epsilon^2 u_{z2}(r,\varphi,\tau)+\dots,\\ p &= \tfrac{1}{8}r^2+\epsilon p_1(r,\varphi,\tau)+\epsilon^2 p_2(r,\varphi,\tau)+\dots, \end{aligned}$$

where $\tau = \epsilon t$ and $\varphi = \theta - \frac{1}{2}t + \epsilon z$, a rotating helical coordinate. The basic flow is simply rigid rotation having axial vorticity unity, and the coordinate φ allows us to move into a frame of reference rotating with this flow. Note that, for any function $f(r,\theta, z,t)$, we have

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial \varphi}, \quad \frac{\partial f}{\partial t} = \epsilon\frac{\partial f}{\partial \tau} - \frac{1}{2}\frac{\partial f}{\partial \varphi} \quad \text{and} \quad \frac{\partial f}{\partial z} = \epsilon \frac{\partial f}{\partial \varphi}.$$

Hence, the combination $\partial f/\partial t + r^{-1}u_\theta \partial f/\partial\theta$ appearing in (34*a–c*) is equal to $\epsilon(\partial f/\partial \tau + r^{-1}u_{\theta 1}\partial f/\partial\varphi)+O(\epsilon^2)$. At $O(\epsilon)$, then, (34*a–d*) gives

$$-u_{\theta 1} + \frac{\partial p_1}{\partial r} = 0, \quad u_{r1} + \frac{1}{r}\frac{\partial p_1}{\partial \varphi} = 0, \quad 0 = 0, \quad \frac{\partial (r u_{r1})}{\partial r} + \frac{\partial u_{\theta 1}}{\partial \varphi} = 0,$$

or simply

$$u_{r1} = -\frac{1}{r}\frac{\partial p_1}{\partial\varphi}, \quad u_{\theta 1} = \frac{\partial p_1}{\partial r}.$$

At $O(\epsilon^2)$, (34*a–d*) gives

$$\frac{\partial u_{r1}}{\partial \tau} + u_{r1}\frac{\partial u_{r1}}{\partial r} + \frac{u_{\theta 1}}{r}\frac{\partial u_{r1}}{\partial \varphi} - \frac{u_{\theta 1}^2}{r} - u_{\theta 2} + \frac{\partial p_2}{\partial r} = 0, \tag{35a}$$

$$\frac{\partial u_{\theta 1}}{\partial \tau} + u_{r1}\frac{\partial u_{\theta 1}}{\partial r} + \frac{u_{\theta 1}}{r}\frac{\partial u_{\theta 1}}{\partial \varphi} + \frac{u_{r1}u_{\theta 1}}{r} + u_{r2} + \frac{1}{r}\frac{\partial p_2}{\partial \varphi} = 0, \tag{35b}$$

$$\frac{\partial u_{z1}}{\partial \tau} + u_{r1}\frac{\partial u_{z1}}{\partial r} + \frac{u_{\theta 1}}{r}\frac{\partial u_{z1}}{\partial \varphi} + \frac{\partial p_1}{\partial \varphi} = 0, \tag{35c}$$

$$\frac{\partial (r u_{r2})}{\partial r} + \frac{\partial u_{\theta 2}}{\partial \varphi} + r\frac{\partial u_{z1}}{\partial \varphi} = 0. \tag{35d}$$

Solving for $u_{\theta 2}$ and u_{r2} from (35*a*) and (35*b*), and substituting into (35*d*), we get the standard two-dimensional vorticity equation with an important additional term representing the effects of vortex stretching. Define

$$\psi \equiv p_1, \quad \zeta \equiv r^{-1}\frac{\partial (r u_{\theta 1})}{\partial r} - r^{-1}\frac{\partial u_{r1}}{\partial \varphi} = \nabla^2\psi, \quad w \equiv u_{z1}.$$

Then, the axial vorticity equation and the axial momentum equation (35*c*) provide a consistent set of reduced equations for the $O(\epsilon)$ fields:

$$\frac{\partial \zeta}{\partial \tau} + J(\psi,\zeta) = \frac{\partial w}{\partial \varphi}, \tag{36a}$$

$$\frac{\partial w}{\partial \tau} + J(\psi, w) = -\frac{\partial \psi}{\partial \varphi}, \tag{36b}$$

where $J(f,g) \equiv r^{-1}(\partial f/\partial r\, \partial g/\partial\varphi - \partial g/\partial r\, \partial f/\partial \varphi)$.

The weak pitch form of the Navier–Stokes equations follows almost immediately. If ν is the coefficient of viscosity, and if we define $D = \nu/\epsilon$, we find the right-hand sides of (36*a*) and (36*b*) are supplemented by the terms $D\nabla^2\zeta$ and $D\nabla^2 w$, respectively.

We next examine the boundary conditions (for the inviscid problem). At the origin, the radial velocity must be finite, so, at $O(\epsilon)$, $\partial\psi/\partial\varphi$ must be zero there. At the external boundary, supposing at first that it is a rigid circular cylinder, we would have $\partial\psi/\partial\varphi = 0$ there also. These two boundary conditions are sufficient to invert Laplace's equation for ψ in terms of ζ.

But suppose the external boundary is free; i.e. it separates rotational interior fluid from irrotational exterior fluid. We then have that each velocity component must be continuous across the boundary $r = R(\varphi, \tau)$, and that fluid particles along the boundary must move with the local fluid velocity there.

The outer fields are expanded in the same way as the inner fields, except u_θ starts with $\frac{1}{2}r^{-1}$ and p starts with $\frac{1}{4}-\frac{1}{8}r^{-2}$. The continuity equation at $O(\epsilon)$ implies that

$$u_{r1} = -\frac{1}{r}\frac{\partial\psi}{\partial\varphi}, \quad u_{\theta 1} = \frac{\partial\psi}{\partial r}$$

for some as yet undetermined function ψ. Irrotationality gives three conditions, one for each component of the vorticity. The vertical component gives $\nabla^2\psi = 0$, at $O(\epsilon)$, while the azimuthal and axial components give $\partial u_{z1}/\partial r = 0$ and $\partial u_{z1}/\partial\varphi = 0$. Hence, u_{z1} is uniform in the external region. Next, we ensure continuity of the velocity fields across the boundary. We assume that the boundary has the expansion

$$R = 1+\epsilon R_1(\varphi,\tau)+\epsilon^2 R_2(\varphi,\tau)+\dots. \tag{37}$$

Then continuity of u_r, at $O(\epsilon)$, requires continuity of $\partial\psi/\partial\varphi$ at $r = 1$. Continuity of u_θ, being careful to note that the basic flow differs in each region, requires

$$R_1(\varphi,\tau)+\frac{\partial\psi^{(\mathrm{in})}}{\partial r}(1,\varphi,\tau) = \frac{\partial\psi^{(\mathrm{out})}}{\partial r}(1,\varphi,\tau). \tag{38}$$

Finally, continuity of u_z requires that the interior field $u_{z1} = w$ be independent of φ along $r = 1$. The last condition to employ is the dynamical one:

$$\frac{\mathrm{D}}{\mathrm{D}t}(r-R(\varphi,\tau)) = 0.$$

Here, $$\mathrm{D}r/\mathrm{D}t = u_r = \epsilon u_{r1}+O(\epsilon^2).$$

However, $$\mathrm{D}R/\mathrm{D}t = \epsilon^2(\partial R_1/\partial\tau+r^{-1}u_{\theta 1}\,\partial R_1/\partial\varphi)+O(\epsilon^3);$$

therefore, at $O(\epsilon)$, the dynamical condition requires that the radial velocity *vanish* at $r = 1$, that is, $\partial\psi/\partial\varphi(1,\varphi,\tau) = 0$. Here, ψ refers to either $\psi^{(\mathrm{in})}$ or $\psi^{(\mathrm{out})}$. Immediately, we can then conclude that the boundary conditions for the interior problem, at $O(\epsilon)$, are the same as if the boundary were a rigid circular cylinder.

But we can go further by obtaining, diagnostically, the boundary deformation at $O(\epsilon)$, i.e. R_1. Consider the solution to Laplace's equation in the exterior. It has the general form

$$\psi = \sum_{m=0}^{\infty} \hat{\psi}_m(\tau)\, r^{-m}\, \mathrm{e}^{\mathrm{i}m\varphi}+\mathrm{c.c},$$

where the $\hat{\psi}_m$ are complex. But the boundary condition that $\partial\psi/\partial\varphi = 0$ at $r = 1$ shows that all of the $\hat{\psi}_m$ are zero except, trivially, $\hat{\psi}_0$. Hence, $\partial\psi/\partial r = 0$ at $r = 1$ as well, and indeed ψ is uniform throughout the external region. We can then use (38) to determine R_1 in terms of the interior flow $\psi^{(\text{in})} = \psi$:

$$R_1(\varphi,\tau) = -\frac{\partial \psi}{\partial r}(1,\varphi,\tau). \tag{39}$$

In other words, $R_1(\varphi,\tau) = -u_{\theta 1}(1,\varphi,\tau)$.

A numerical code was written following the guidelines of Landman (1990), who developed a code for the full helical equations. The code uses second-order finite differences in the radial direction, with equally spaced radii, and a finite Fourier expansion in the azimuthal direction. There are 129 radial intervals and 129 azimuthal wavenumbers, the latter ranging from -64 to 64. Nonlinear products are calculated in physical space, with the aid of fast Fourier transforms to exchange spectral fields and real fields. A small amount of diffusion is included purely to stabilize the code – the no-slip boundary condition at $r = 1$ is not imposed. In its place, we require that the second radial derivative of all fields vanish at $r = 1$. The coefficient of diffusion D is set to $2(\Delta r)^2$, where $\Delta r = \frac{1}{129}$. The time integration begins with a second-order Runga–Kutta step and continues with second-order Adams–Bashforth steps. However, if the peak vorticity doubles some prescribed value, the time step is halved, and the calculation restarts with a Runge–Kutta step. Each successive vorticity doubling is dealt with in the same way. The initial time step is $\Delta\tau = 0.001$.

The exact, steadily-rotating, non-axisymmetric solutions provide a convenient check on the accuracy of the code. With the type of diffusion employed, these solutions simply slowly decay while retaining the same spatial structure and rotational frequency. A test calculation beginning with the $m = 1$ solution ($\alpha = +j_{11}$), for instance, proves to be indistinguishable from the exact solution up to $\tau = 25$.

We turn next to the evolution of superposed solutions. The notation $\mathscr{A}_1 \hat{\zeta}_m^{\pm} + \mathscr{A}_2 \hat{\zeta}_n^{\pm}$ is used to denote the initial condition

$$\begin{aligned}\zeta(r,\varphi,0) &= j_{m1}\mathscr{A}_1 J_m(j_{m1} r)\,\mathrm{e}^{\mathrm{i}m\varphi} + j_{n1}\mathscr{A}_2 J_n(j_{n1} r)\,\mathrm{e}^{\mathrm{i}n\varphi} + \text{c.c.},\\ w(r,\varphi,0) &= \pm\mathscr{A}_1 J_m(j_{m1} r)\,\mathrm{e}^{\mathrm{i}m\varphi} \pm \mathscr{A}_2 J_n(j_{n1} r)\,\mathrm{e}^{\mathrm{i}n\varphi} + \text{c.c.},\end{aligned}$$

where $\mathscr{A}_1$ and $\mathscr{A}_2$ are complex constants in general. Figure 1 shows the evolution of $0.209\hat{\zeta}_1^+ + 0.052\hat{\zeta}_1^-$. By $\tau = 25$, the vorticity has begun to intensify markedly. The intensification occurs predominantly in regions of large axial velocity gradients ($\partial w/\partial\varphi$ fuels ζ), and the vorticity takes on a multiple sheet-like structure. The eventual arrest of vorticity growth in the thin structures is largely attributable to inviscid mechanisms, with diffusion playing only a minor role. Note that w itself does not intensify significantly. This is because $w - \frac{1}{2}r^2$ is materially conserved (in the absence of diffusion). Numerous other calculations exhibit the same sequence of events: an initial twisting of the ζ and w fields, followed by an intensification of ζ along gradients of w, with thin, rapidly intensifying, closely spaced ridges and troughs of vorticity. A second example, having a qualitatively different evolution, is given in figure 2, for $-0.5\hat{\zeta}_0^+ + 0.039\hat{\zeta}_2^-$ (note: $\hat{\zeta}_0^+$ is an axisymmetric flow). At first, two symmetrical centres of vorticity develop; then these rapidly intensify and thin, forming spiralling sheet-like structures. Unlike in the previous example, the axial velocity field manages to develop near discontinuities and in this way fuels an explosive growth in vorticity. The growth in peak vorticity is very great, being super-exponential at late times.

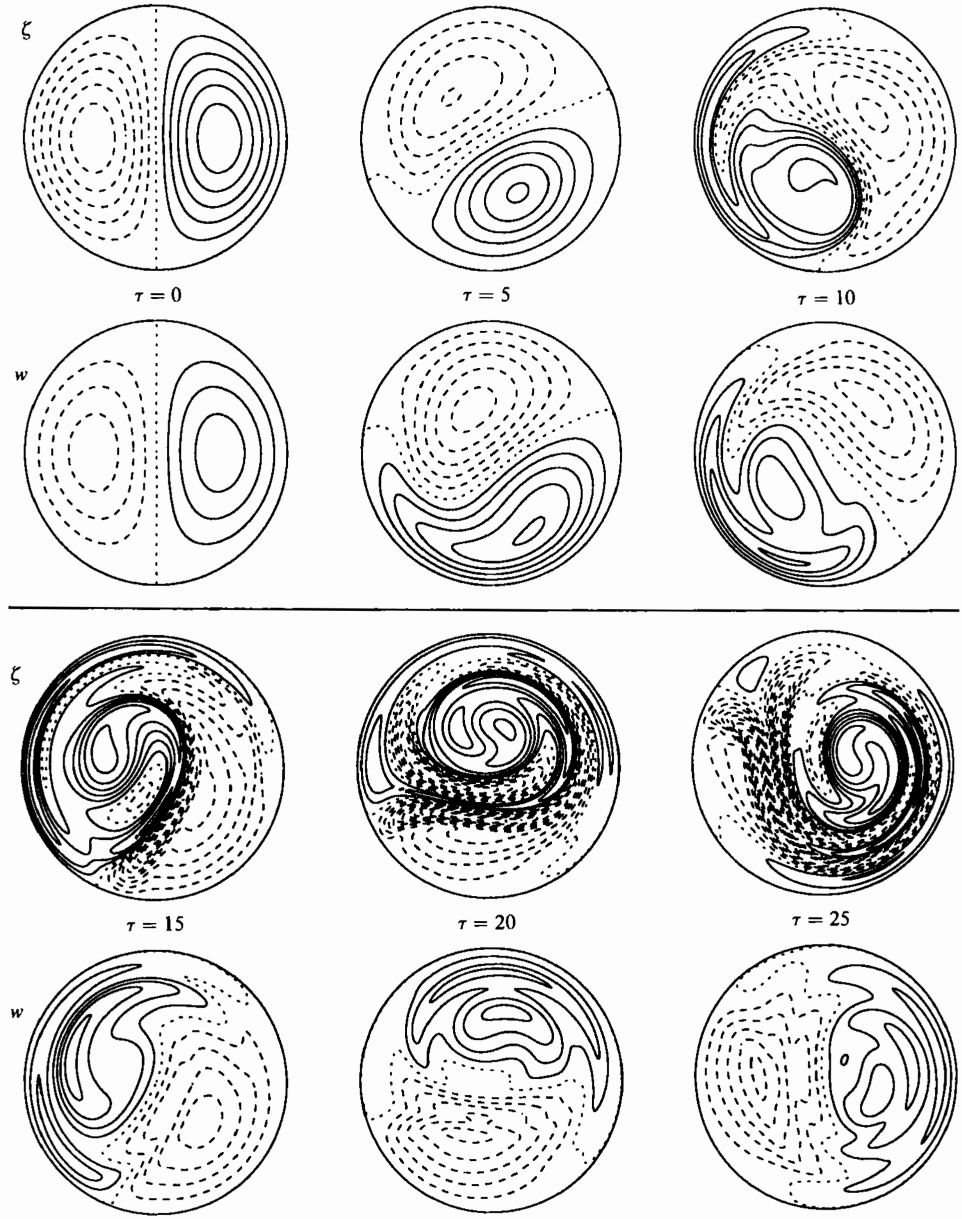

FIGURE 1. The evolution of $0.209\hat{\zeta}_1^+ + 0.052\hat{\zeta}_1^-$. Time increases to the right. The top frames show ζ and the bottom frames w, contoured at intervals of 0.2 and 0.05, respectively. Positive values are contoured with solid lines, negative values with dashed lines, and the zero contour with dotted lines.

5. Discussion

This paper has shown that a helically symmetric flow composed of a basic part in rigid rotation and a disturbance part whose vorticity is proportional to its velocity satisfies a simple, linear eigen-equation. Solutions to this equation exactly satisfy the three-dimensional, nonlinear Euler equations and are therefore valid for arbitrary amplitude. Remarkably, the solutions for flow in a straight pipe of circular cross-

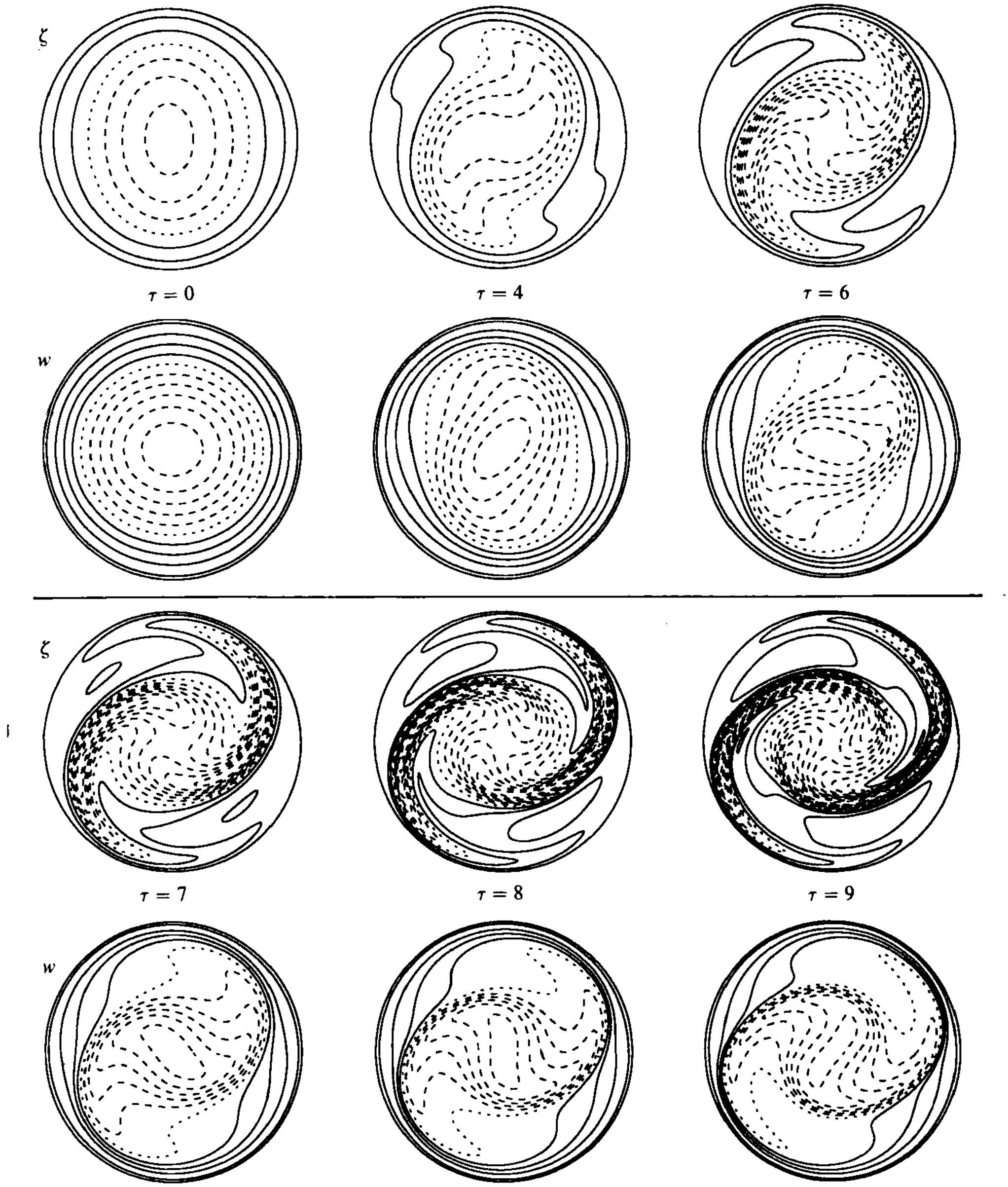

FIGURE 2. The evolution of $-0.5\hat{\zeta}_0^+ + 0.039\hat{\zeta}_2^-$. The contour intervals in ζ and w are 0.4 and 0.1 respectively. The average values of these fields have been subtracted, but this does not affect the dynamics.

section turn out to be the eigenfunctions worked out by Kelvin (1880) in his *linear* stability analysis of rigid rotation.

The disturbance vorticity and velocity need not even be parallel as long as the cross-product of these two fields is proportional to the gradient of the disturbance streamfunction. The resulting eigen-equation is then supplemented by an inhomogeneous term.

Parallel results hold for MHD. In addition to a basic rigid rotation, there may exist a uniform current density directed along the axis of rotation. The solutions to the linear MHD equation have precisely the same spatial structure as found for the linear Euler equation. The magnetic field simply alters the frequencies of the solutions.

These exact solutions appear to be distinct from those constructed by Craik (1988), whose solutions take the form of plane waves with simple spatial structure, and from those constructed by Squire (1956), whose axisymmetric solutions do not possess helical symmetry. They also provide examples of flows with non-trivial topology (see Moffatt 1989 for general remarks). Viscous effects cannot be easily included, although it may be possible to patch these solutions into boundary layers for small viscosity.

Numerical calculations indicate that many of these solutions are unstable, in the full, nonlinear sense, at least in the limit of small pitch. In real flows, these solutions may therefore never occur, or only appear in transition from one unsteady flow to another. On the other hand, effects which have not been included here, for example finite pitch or stratification, may bring about stability. Finite pitch has the effect of bringing into force additional terms in the vorticity equation (see (B 1*b*) in Appendix B) which did not enter into the weakly nonlinear equations modelled in §4, and these additional terms may lead to significantly different flow structures, if not stability. The effects of weak axial stratification may also bring about stabilization, since buoyancy tends to suppress axial motions. These possibilities are under active investigation.

Another idea is to examine the special subclass of helical flows characterized by constant v in the non-rotating frame of reference. Such flows materially conserve the quantity $q \equiv h^2\zeta$ (as one can verify from (B1a,b) in Appendix B). Thus, if q were initially piecewise-constant in a flow, it would remain piecewise-constant, and furthermore, the boundaries across which q jumps would uniquely determine the velocity field everywhere. It is thus possible to develop a Lagrangian model for the motion of the q-discontinuity contours, thereby enabling one to examine the evolution of complex, nonlinear helical flows. In two extreme cases, this model has already seen extensive application: in the limit of zero pitch (two-dimensional flow), the model reduces to 'contour dynamics' (Zabusky, Hughes & Roberts 1979; Dritschel 1989 and references therein); in the limit of infinite pitch (axisymmetric flow), a modified form of contour dynamics also results (Pozrikidis 1986; Shariff, Leonard & Ferzinger 1989). In general, the model relies on being able to find Greeen's function for the operator $h^2\mathscr{L}$ (see (4)). For two-dimensional flow, Green's function reduces to the logarithm of the distance between two points, while for axisymmetric flow, it reduces to a combination of complete elliptic integrals of the first and second kind (Shariff *et al.* 1989). For the intermediate range of pitch, it has not yet been possible to find a closed-form expression for Green's function, and there is some doubt that one exists. In any case, there is a wide range of phenomena open to exploration using this model, such as steadily rotating nonlinear waves on a vortex column, or multiple, corrotating vortices; the stability of these equilibria both to irrotational and rotational disturbances (non-uniform v for example); and the nonlinear dynamics of interacting vortices, just to name a few possibilities.

The instability depicted in figure 2 may have application to a certain curious phenomenon sometimes observed in atmospheric vortices. It is observed that tornadoes can break up into multiple vortices, helically intertwined about a common centre of rotation and of significantly greater intensity than the parent tornado (see, for example, Fujita 1970; Agee *et al.* 1977). Experimental work has also captured what appears to be the same phenomenon (Ward 1972; Church *et al.* 1979). Yet, a complete understanding of this phenomenon is still outstanding.

One idea is that multiple vortices are preceded by the formation of a 'breakdown bubble', a transition along the parent vortex column from supercritical to subcritical flow (see Benjamin 1962 for a precise definition of these terms). Observational evidence for this is noted by Pauley & Snow (1987) and Lugt (1989). The flow feeding into a tornado near the ground is typically supercritical, often turning to subcritical further up the vortex column (or the entire vortex column may become subcritical in cases when the breakdown bubble reaches the ground). If we consider Benjamin's (1962) theory of vortex breakdown, for the special case of a supercritical Rankine flow (rigid rotation and uniform axial velocity), then there exists a family of associated subcritical columnar vortex flows with non-uniform axial velocity parameterized by the closeness of the supercritical flow to criticality. The case of a marginally supercritical flow formed the example illustrated in figure 2. We can conclude that the associated subcritical flow is unstable to non-axisymmetric disturbances, and, importantly, develops structures reminiscent of multiple vortices in tornadoes. The numerical and observational results may be brought into closer correspondence, it is believed, by using the full helical equations (finite pitch) and weak stratification. A more complete account will be forthcoming in a future paper.

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Appendix A. More general helical domains

In this Appendix, boundary conditions are derived for domains asymmetrically distributed about the axis of rotation, yet retaining a three-dimensional helical structure. These allow one to construct exact nonlinear solutions in curved pipes, e.g. an infinitely extended corkscrew, or a pipe with an elliptical cross-section.

First, it is necessary to adopt a frame of reference in which both the domain boundary and the enclosed flow appear steady. The rotation rate of this frame of reference, say Ω, generally differs from the rotation rate of the basic flow, Ω_0. To find Ω in terms of Ω_0, we must manipulate the momentum and vorticity equations (8*a*, *b*) as in §2 to produce a linear evolution equation like (10). However, (8*a*, *b*) apply only when $\Omega = \Omega_0$, but they can be amended easily by replacing $\boldsymbol{u}$ by $\boldsymbol{u}+(\Omega_0-\Omega)\, r\boldsymbol{e}_\theta$ and $\boldsymbol{\omega}$ by $\boldsymbol{\omega}+2(\Omega_0-\Omega)\,\boldsymbol{e}_z$. This has the ultimate effect of altering (10) to

$$\frac{\partial\psi}{\partial t}+\left[\Omega_0-\Omega+\frac{2\epsilon\Omega_0}{\alpha}\right]\frac{\partial\psi}{\partial\phi}=0; \qquad \text{(A 1)}$$

thus, the solution will be steady if we choose

$$\Omega=\Omega_0\left(1+\frac{2\epsilon}{\alpha}\right). \qquad \text{(A 2)}$$

Now, in this frame of reference, there will be a residual rotation equal to $\Omega_0-\Omega=-2\epsilon\Omega_0/\alpha$ which will generally have a non-zero normal component on the boundary. This will force an inhomogeneous boundary condition for ψ since the total normal component of the flow must vanish.

Suppose that the boundary is given by $r=R(\phi)$. Then the radial velocity of a particle on the boundary must be directly compensated by the radial flow there, or

$$\frac{\mathrm{D}R}{\mathrm{D}t}=u_r(R,\phi). \qquad \text{(A 3)}$$

Expanding the material derivative of R, in the rotating frame of reference, and using the definition of u_r from (11a), (A 3) may be written

$$-\Omega\frac{\mathrm{d}R}{\mathrm{d}\phi}+\left(\Omega_0+\frac{\mathrm{D}\phi}{\mathrm{D}t}\right)\frac{\mathrm{d}R}{\mathrm{d}\phi}=-\frac{1}{R}\frac{\partial\psi}{\partial\phi}. \tag{A 4}$$

Now, since $\phi=\theta+\epsilon z$, then $\mathrm{D}\phi/\mathrm{D}t=r^{-1}u_0+\epsilon u_z=r^{-1}\,\partial\psi/\partial r$. Hence, (A 4) can be rewritten

$$\frac{\partial\psi}{\partial\phi}+\left(\frac{\partial\psi}{\partial r}-\frac{2\epsilon\Omega_0}{\alpha}R\right)\frac{\mathrm{d}R}{\mathrm{d}\phi}=0. \tag{A 5}$$

If the origin is included in the domain, $\partial\psi/\partial\phi=0$ there as before.

Appendix B. More general basic flows

In this appendix, it is proved that no other basic flow reduces the helically symmetric Euler equations to a linear equation. We must now consider the full nonlinear helical equations (see Landman 1990 and references therein). These are obtained, after a considerable amount of algebra, by taking the scalar product of (8a) and (8b) with $\boldsymbol{h}$ and using (3a) and (3b). Suppose for the moment that $\boldsymbol{u}$ includes both the basic flow and the disturbance. Then, in a non-rotating frame of reference, the helical equations are

$$\frac{\partial v}{\partial t}+J(\psi,v)=0, \tag{B 1a}$$

$$\frac{\partial\zeta}{\partial t}+J(\psi,\zeta)=2\epsilon h^2\left[J(\psi,v)-\epsilon\left(v\frac{\partial v}{\partial\phi}+\zeta\frac{\partial\psi}{\partial\phi}\right)\right], \tag{B 1b}$$

where
$$J(f,g)\equiv h^{-2}\boldsymbol{h}\cdot(\boldsymbol{\nabla}f\times\boldsymbol{\nabla}g)=\frac{1}{r}\left(\frac{\partial f}{\partial r}\frac{\partial g}{\partial\phi}-\frac{\partial f}{\partial\phi}\frac{\partial g}{\partial r}\right). \tag{B 1c}$$

The effect of a background rigid rotation in a frame of reference rotating with this basic flow is to add the terms $-2\epsilon\Omega_0\,\partial\psi/\partial\phi$ and $2\epsilon\Omega_0\,\partial v/\partial\phi$ to the right-hand sides of (B 1a) and (B 1b), respectively. So, for this basic flow, we can immediately see that $v=\alpha\psi$ and $\zeta=-\alpha^2\psi+\pi'/h^2$ collapse (B 1a) and (B 1b) into a single linear equation for ψ, namely (10).

In the more general case, we suppose the flow to be divided into an imposed basic part $\boldsymbol{U}(r,\phi,t)$ and a disturbance part $\boldsymbol{u}(r,\phi,t)$, the latter satisfying the extended Beltrami constraint $\boldsymbol{\omega}\times\boldsymbol{u}=-\pi'\boldsymbol{\nabla}\psi$. In the helical equations above, we replace ψ by $\Psi(r,\phi,t)+\psi(r,\phi,t)$, v by $V(r,\phi,t)+\alpha\psi(r,\phi,t)$ and ζ by $Z(r,\phi,t)-\alpha^2\psi(r,\phi,t)+\pi'/h^2$. Here, V and Z are the imposed basic flow quantities (they are linked to Ψ through the obvious analogue of (4)), ψ is the disturbance stream function, and the extended Beltrami constraint on the disturbance has already been used. Inserting these fields into (B 1a) and (B 1b), we find after further multiplying (B 1a) by $-\alpha$

$$-\alpha^2\frac{\partial\psi}{\partial t}-\alpha J(\psi,V-\alpha\Psi)=\alpha\left(\frac{\partial V}{\partial t}+J(\Psi,V)\right), \tag{B 2a}$$

$$\begin{aligned}-\alpha^2\frac{\partial\psi}{\partial t}+J(\psi,Z+\alpha^2\Psi)&=2\epsilon h^2\left(J(\psi,V-\alpha\Psi)-\epsilon(Z+\alpha V)\frac{\partial\psi}{\partial\phi}-\epsilon\alpha\psi\frac{\partial}{\partial\phi}(V-\alpha\Psi)\right)\\&\quad-\left\{\frac{\partial Z}{\partial t}+J(\Psi,Z)-2\epsilon h^2\left[J(\Psi,V)-\epsilon\left(V\frac{\partial V}{\partial\phi}+Z\frac{\partial\Psi}{\partial\phi}\right)\right]\right\}.\end{aligned} \tag{B 2b}$$

Both are equations for the evolution of ψ. To be consistent, they must be identical, and this requires the coefficients of $\partial\psi/\partial r$, $\partial\psi/\partial\phi$, ψ, and 1 to match, or

$$\alpha\frac{\partial}{\partial\phi}(V-\alpha\Psi)+\frac{\partial}{\partial\phi}(Z+\alpha^2\Psi)-2\epsilon h^2\frac{\partial}{\partial\phi}(V-\alpha\Psi)=0, \tag{B 3a}$$

$$\alpha\frac{\partial}{\partial r}(V-\alpha\Psi)+\frac{\partial}{\partial r}(Z+\alpha^2\Psi)-2\epsilon h^2\left(\frac{\partial}{\partial r}(V-\alpha\Psi)+\epsilon r(Z+\alpha V)\right)=0, \tag{B 3b}$$

$$\alpha\frac{\partial}{\partial\phi}(V-\alpha\Psi)=0, \tag{B 3c}$$

$$\frac{\partial}{\partial t}(Z+\alpha V)+J(\Psi,Z+\alpha V)-2\epsilon h^2\left[J(\Psi,V)-\epsilon\left(V\frac{\partial V}{\partial\phi}+Z\frac{\partial\Psi}{\partial\phi}\right)\right]. \tag{B 3d}$$

Equation (B 3c) implies that $V-\alpha\Psi$ is independent of ϕ which, in conjunction with (B 3a), implies that $Z+\alpha^2\Psi$ is also independent of ϕ. Let, therefore,

$$A(r,t)=V-\alpha\Psi,\quad B(r,t)=Z+\alpha V \tag{B 4a, b}$$

(note $Z+\alpha^2\Psi=B-\alpha A$). Expressed another way,

$$V=A+\alpha\Psi,\quad Z=(B-\alpha A-\pi'/h^2)-\alpha^2\Psi+\pi'/h^2,$$

which is to say that although V and Z can depend on ϕ, they can do so only by way of an extended Beltrami flow having the same constant of proportionality α. Since ψ is already assumed to be such a flow, we can incorporate the ϕ-varying part of Ψ into ψ and so consider Ψ, and hence V and Z, independent of ϕ.

Now (B 3b) gives

$$\frac{\partial B}{\partial r}-2\epsilon h^2\left(\frac{\partial A}{\partial r}+\epsilon rB\right)=0, \tag{B 5}$$

and with this, (B 3d) gives simply

$$\frac{\partial B}{\partial t}=0.$$

Hence, from (B 5), $\partial A/\partial r$ is also independent of time, and in fact it is sufficient to choose A as a function of r alone. Equation (B 5) gives a non-trivial constraint on the basic flow, now only a function of radius r. It can be rewritten as

$$\frac{\mathrm{d}h^2B}{\mathrm{d}r}=2\epsilon h^4\frac{\mathrm{d}A}{\mathrm{d}r}, \tag{B 6}$$

and when combined with the analogue of (4) for the basic flow,

$$\frac{1}{rh^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(rh^2\frac{\mathrm{d}\Psi}{\mathrm{d}r}\right)=Z+2\epsilon h^2V, \tag{B 7}$$

we see from the definitions $A=V-\alpha\Psi$ and $B=Z+\alpha V$ that we are apparently free to choose one of the fields, say $\Psi(r)$, and determine the other two from (B 6) and (B 7). For example, rigid rotation corresponds to

$$\Psi=\tfrac{1}{2}\Omega_0r^2;\quad V=-\epsilon\Omega_0r^2;\quad Z=2\Omega_0. \tag{B 8}$$

For arbitrary $\Psi(r)$, however, the disturbance evolution equation, (B 2*a*) or (B 2*b*), reduces to

$$\frac{\partial \psi}{\partial t}=\frac{1}{\alpha r}\frac{\mathrm{d}A}{\mathrm{d}r}\frac{\partial \psi}{\partial \phi}, \tag{B 9}$$

whose general solution is $\psi(r, \phi-\alpha^{-1}(r^{-1}\,\mathrm{d}A/\mathrm{d}r)\,t)$. If this is inserted into (7), we find that an explicit dependence on t cannot be eliminated, unless $r^{-1}\,\mathrm{d}A/\mathrm{d}r$ is constant. But then the basic flow is just rigid rotation. Hence, rigid rotation is the only basic flow that is consistent with linear disturbance evolution.

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